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Interacting Matter and Radiation in Homogeneous Isotropic World Models

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Abstract

Transitions between two or more fluids may result in changes of the total amount of radiation in the universe without violating overall energy conservation. Special rate equations for such processes are discussed, for which the radius of the universe as a function of time can be found by numerical integration of a single differential equation, and the radiation density can then be easily obtained, although it must be verified as positive for all times. The integrations are carried out and reported here for the simplest rate equation for two fluids for various values of the initial mass of one fluid, the transition rate, and the cosmological constant.

1. *Introduction*

The early interest in the problem of interacting matter and radiation in cosmology shows a desire to reconcile thermodynamic predictions on the ultimate fate of the universe with the seemingly unrelated predictions of cosmology (Tolman, 1931a, 1931b, 1932, 1934). This question is still important today (Pegg, 1972), but modern research is more concerned with the evolution of the universe from the still unknown initial state. This deterministic approach to cosmology is so much more promising today than at earlier times, because of our greatly increased knowledge of elementary particles and their interactions. However, particle physics, which derives its data from studies over comparatively short time intervals, cannot give information on persistent trends over cosmic time scales and may therefore be unable to explain the evolution of the universe completely. We offer here some estimates of the cumulative effects of some processes which are consistent with energy and momentum conservation as well as the conservation laws associated with symmetries within the families of elementary particles. We accept the postulates of isotropy and homogeneity of the universe and use a phenomenological description for two interacting fluids, with one fluid increasing at the expense of the other one and giving rise to a gradually increasing radiation density (McIntosh, 1968; May & McVittie, 1970).

The three components have a common bulk velocity and conservation of momentum is trivially satisfied. Energy is not conserved for each component separately, but the energy of each component may change according to a rate equation which is restricted only by the requirement of overall energy conservation. Since general rate equations lead to several coupled differential equations, we make some simplifying assumptions. First, we assume that the transitions between the matter fluids are independent of the amount of radiation present, i.e. no stimulated transitions do occur. Secondly, we assume that the rate equations contain the derivative of the world radius in such a way that we can solve them explicitly in terms of the world radius. The densities can then be inserted into Einstein's equations, or rather a suitable combination of Einstein's equations which does not contain the radiation density. We obtain then the radius of the universe as a function of time, and calculate finally the radiation density either from the remaining Einstein equations, or from the energy conservation law. In either case the radiation density may not be positive for all times, and solutions for which it becomes negative must be rejected. It should perhaps be pointed out that the time variation of the densities of the fluids gives a physical meaning to the concept of the age of the universe. If only matter fluids were present and no transitions occurred, the age of the universe would have only geometrical meaning, since the fluids would expand and contract together according to the cosmological solution. Of course, the age of the real universe is determined by a great number of parameters, viz. the relative frequencies of all elementary particles and their antiparticles. But the models presented in this paper are probably the simplest models which conform to the demand that the age of the universe is related to its physical features.

2. *The Rate Equations*

Basic to the theory is the conservation of energy and momentum, expressed in

$$(\sqrt{(-g)}\,\rho V^{\alpha})_{,\alpha} + p(\sqrt{(-g)}\,V^{\alpha})_{,\alpha} = 0 \tag{2.1}$$

and the hydrodynamic equations. The latter need not be considered if the various fluids have one common bulk velocity; and a co-moving coordinate system with $V^0 = 1$, $V^1 = V^2 = V^3 = 0$ can be used. We may decompose in (2.1) the total energy density

$$\rho = \rho_1 + \rho_2 + u \tag{2.2}$$

and, accordingly, the total pressure

$$p = p_1 + p_2 + \tfrac{1}{3}u. \tag{2.3}$$

The interactions which lead to non-conservation of the individual energies can be described by rate equations for the fluids $i = 1$ and 2

$$(\sqrt{(-g)}\,\rho_i V^\alpha)_{,\alpha} + p_i(\sqrt{(-g)}\,V^\alpha)_{,\alpha} = f_i \tag{2.4}$$

with

$$\sum f_i + (\sqrt{(-g)}\,uV^\alpha)_{,\alpha} + \tfrac{1}{3}u(\sqrt{(-g)}\,V^\alpha)_{,\alpha} = 0 \tag{2.5}$$

where the functions f_i could depend on ρ_k and on $\sqrt{-g}$ but, as explained in the introduction, should not depend on u. Equation (2.5) can be considered as a differential equation for u which can be solved when ρ_k and $\sqrt{-g}$ have been determined. We direct now our attention to (2.4) and assume that f_i is a scalar density and write

$$f_i = \sqrt{(-g)}\,A_i + (\sqrt{(-g)}\,V^\alpha)_{,\alpha} B_i \tag{2.6}$$

where A_i and B_i are scalar functions of ρ_k, which do not contain the metric tensor. They describe phenomenologically a fundamental law of interaction and should be form-invariant. On the other hand, the scalar functions ρ_k may depend on x^α explicitly as well as implicitly through the metric tensor $g_{\alpha\beta}$. In the co-moving frame of reference, only differentiation with respect to $x^0 = t$ will occur; and if equations (2.4) should be soluble without prior knowledge of $\sqrt{-g}$, the conditions

$$\frac{\partial \rho_i}{\partial t} = A_i \tag{2.7}$$

$$\sqrt{(-g)}\frac{\partial \rho_i}{\partial\sqrt{-g}} + \rho_i + p_i = B_i \tag{2.8}$$

must be satisfied. We get then the integrability condition

$$\frac{\partial}{\partial\sqrt{-g}}(\sqrt{(-g)}\,A_i) = \frac{\partial}{\partial t}(B_i - p_i) \tag{2.9}$$

and recalling that A_i and B_i are functions of ρ_k only, we can rewrite (2.9)

$$\frac{\partial A_i}{\partial \rho_k}B_k - \frac{\partial B_i}{\partial \rho_k}A_k + A_i + \frac{\partial p_i}{\partial t} = \frac{\partial A_i}{\partial \rho_k}(\rho_k + p_k) \tag{2.10}$$

These conditions define a fairly general class of rate equations from which we will single out a few special cases. The numerical integration in the next section is based on the choice

$$A_1 = -\alpha\rho_1, \qquad B_1 = 0, \qquad p_1 = p_2 = 0 \tag{2.11}$$

which describes the gradual decrease of the density of the first fluid. If the first fluid is identified with nucleons and the second fluid describes a complex nucleus with atomic weight A, we must observe the conservation law for the number of nucleons. Introducing the number densities n_1 and n_2, the nucleon mass m and the binding energy ε of the second fluid, we have

$$\rho_1 = n_1 mc^2, \qquad \rho_2 = n_2(Amc^2 - \varepsilon) \tag{2.12}$$

and since the total nucleon number $(n_1 + n_2 A)$ should remain constant, we must demand that

$$\rho_1(1 - \varepsilon/Amc^2) + \rho_2 \tag{2.13}$$

remains constant. We will neglect the pressure of the matter fluids and set in accordance with (2.13)

$$A_2 = \alpha(1 - \varepsilon/Amc^2)\,\rho_1 \tag{2.14}$$

This is a simple model for the process of fusion of hydrogen into some complex nucleus and if the sign of α is reversed the inverse process of decomposition is described. Such interaction terms are equally possible with the B_i terms instead of the A_i terms. Equation (2.7) shows that ρ_k does not depend explicitly on time and the solution follows from (2.8)

$$\rho_i = \text{const.}\,(-g)^{1/2(1+\alpha)} \tag{2.15}$$

as compared with the solution in the previous case (2.11)

$$\rho_i = \text{const.}\,(-g)^{-1/2}\,\mathrm{e}^{-\alpha t} \tag{2.16}$$

The rate equation with $A_i = 0$ can be treated easily without the arguments leading to (2.7) and (2.8). In the co-moving frame $\sqrt{(g_{00})}\,V^0 = 1$ and by transformation to a new time coordinate

$$\sqrt{(-g_{11}g_{22}g_{33})} \sim \mathrm{e}^{-\Omega} \tag{2.17}$$

one obtains a simple linear equation for ρ_k. This is the approach used by Hughston & Shepley (1970). We consider finally general linear forms of A_i and B_i in ρ_k:

$$A_i = \alpha_{ik}\rho_k, \qquad B_i = \beta_{ik}\rho_k \tag{2.18}$$

and neglect again the pressure. Equation (2.10) leads to the conditions

$$(\alpha_{ik}\beta_{kl} - \beta_{ik}\alpha_{kl})\,\rho_l = 0 \tag{2.19}$$

and either one of the matrices is the unit matrix, or both are diagonal. We find from (2.7)

$$\rho_i = C_i\,\mathrm{e}^{\alpha_1 t} + D_i\,\mathrm{e}^{\alpha_2 t} \tag{2.20}$$

where α_1 and α_2 are the roots of a secular equation and C_i and D_i are the corresponding solutions. They will be functions of $\sqrt{-g}$ and can be determined from (2.8). It can be seen by inspection that these functions, and therefore also ρ_i, will be inversely proportional to $\sqrt{-g}$ with a rather complicated power. The ultimate justification of any rate equation must come from a basic kinetic theory (Ehlers, 1971; Stewart, 1971), and one might expect quadratic expressions in the densities in analogy to Boltzmann's equation.

3. *World Radius and Radiation Density*

We seek to eliminate the radiation density u from Einstein's equation and write for the source term

$$T^{\alpha\beta} + \tfrac{4}{3}uV^{\alpha}V^{\beta} - \tfrac{1}{3}ug^{\alpha\beta} \tag{3.1}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor of the matter fluids. Quite generally, contraction with $V_{\alpha}V_{\beta}$ gives an expression for u which can be inserted in Einstein's equation. It must be assumed that the equation of state for matter differs from $p = \frac{1}{3}\rho$, valid for radiation. In the homogeneous, isotropic model, we may simply use the trace of Einstein's equation which is independent of u. In the usual coordinate frame

$$g_{00} = 1, \qquad g_{11} = g_{22} = g_{33} = -\frac{R^2(t)}{(1 + kr^2/4)^2} \tag{3.2}$$

where $R(t)$ is the radius of the universe and $k = -1$, 0 or $+1$, the trace is equivalent to

$$R\ddot{R} + \dot{R}^2 + k = \tfrac{2}{3}\Lambda R^2 + \frac{4\pi}{3}(\rho_1 + \rho_2)R^2 \tag{3.3}$$

Since the numerical calculations were carried out only for the elliptic case $k = 1$, we will consider now only this case. The matter densities follow from (2.11) and (2.14)

$$\rho_1 = M_1/R^3\pi^2 . \mathrm{e}^{-\alpha(t-t_0)} \tag{3.4}$$

$$\rho_2 = M_2/R^3\pi^2 - M_1(1 - \varepsilon/Amc^2)/R^3\pi^2 . (\mathrm{e}^{-\alpha(t-t_0)} - 1) \tag{3.5}$$

The total masses of the fluids at the time t_0 were here denoted with M_1 and M_2 and the volume of the closed universe was assumed to be π^2R^3. The sum of (3.4) and (3.5), which alone enters in (3.3), can be written as

$$\rho_1 + \rho_2 = (M_1 + M_2)/R^3\pi^2 + M_1\eta/R^3\pi^2 . (\mathrm{e}^{-\alpha(t-t_0)} - 1) \tag{3.6}$$

where we have introduced the 'efficiency'

$$\eta = \varepsilon/Amc^2 \tag{3.7}$$

which, besides M_1 and α provides a measure for the degree of the mass depletion. The value for η for the spontaneous fusion of hydrogen into iron would be approximately 0·01, but significant deviations of $R(t)$ from the non-radiating case occur only for somewhat larger values η and we have therefore extended the calculations for values of η up to unity (Knight, 1972). The decay constant α was chosen in the range between 0·01 and 0·1 times 10^{-9} (years)$^{-1}$, with the lower limit corresponding to the mass depletion of our sun, assuming an energy loss of 10^{33} erg/sec (Sciama, 1971), and an efficiency of 0·01. The world radius R with the dimension of a length and measured in units of 10^9 light years was given the initial value 10 at time t_0. The initial slope, determined by Hubble's constant, was taken as 0·7, with the velocity of light having the value unity in the dimensions light years and years. We calculated the world radius for various values of the cosmological constant Λ, mainly to determine the shift of the critical value, which for the non-radiating case is given by

$$\Lambda_c = \pi^2/16M^2 \tag{3.8}$$

We varied Λ between $\pm 0{\cdot}005$ (10^9 light years)$^{-2}$. Outside this range the effect of the constant was so predominant that radiative effects were negligible in comparison. It remains to discuss the values for the masses M_1 and M_2 which were used in our computation. Since the curves $R(t)$ showed little spread in the short time interval $t < t_0$, we avoided the difficult problem of selecting realistic values and neglected M_2 altogether. This implies that the universe contains only the first fluid at time t_0 and, although the density of the second fluid may become considerable in the future $t > t_0$, it has necessarily negative densities in the past $t < t_0$. The initial conditions are quite generally connected by the well-known first-order differential equation (Robertson & Noonan, 1968)

$$3(\dot{R}^2 + 1)/R^2 = \Lambda + 8\pi\rho \tag{3.9}$$

Neglecting here M_2 and therefore ρ_2 at time t_0, and keeping Hubble's constant and the radius of the universe fixed, excessively large values of ρ_1 and M_1 will result in negative values of u. For vanishing cosmological constant, the limiting value of M_1 is 18×10^9 light years, corresponding to a mass density of 3×10^{-30} g/cm^3. Larger values of M_1 will require a negative cosmological constant to ensure a positive radiation density. On the other hand, much smaller values for M_1 would require either larger values for the cosmological constant than one is willing to accept or unreasonably large radiation densities. Reasonable values can be inferred from the known black-body temperature

$$u/\rho_1 \sim 7 \times 10^{-4} \tag{3.10}$$

which justifies the use of the critical value for ρ_1 as derived from (3.9) for $u = 0$.

The figure shows the general trend of the curves $R(t)$ with increasing efficiency η. The more effective the interaction in producing radiation, the faster the collapse of the universe, and we found the same trend to hold as the decay constant α was increased. One may expect the reverse trend for negative values of α, describing a decrease of radiation. The critical value of the cosmological constant, given by (3.8) in the non-radiating case, is accordingly shifted to higher values as η or α are given finite positive values. For $M = 15{\cdot}5$, (3.8) gives $\Lambda_c = 0{\cdot}0025$, yet even for $\Lambda = 0{\cdot}003$ the universe would collapse for the admittedly rather large efficiency $\eta = 1$. On the whole the curves do not differ greatly from the usual curves for non-interacting fluids, although there are considerable quantitative differences in the later stages of the universe. Only for larger M_1, exceeding the critical value derived from (3.9) and leading therefore to negative radiation pressures, do we find a greater sensitivity towards variation of the efficiency. For instance, $M_1 = 31, \Lambda = -0{\cdot}01, \alpha = 0{\cdot}1$ gives for $\eta = 0$ and for $\eta = 0{\cdot}01$ smooth (almost) periodic functions with a minimum radius of the universe of about 3×10^9 light years. But for $\eta = 0{\cdot}1$, we see again the familiar collapse to a point-size universe in about 25×10^9 years. For the same mass and $\Lambda = 0$, the curves for efficiencies between zero and 0·1 are smooth and differentiable, although the periods and the minima differ greatly. Of course, negative radiation

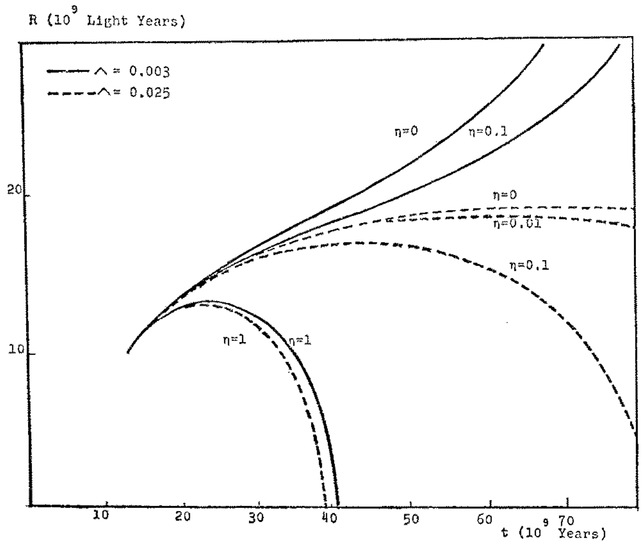

Figure 1—The world radius as a function of time for the mass $M_1 = 15{\cdot}5$ and various values of the cosmological constant and the efficiency η. The decay constant was assumed as $0{\cdot}1 \times 10^{-9}$ (years)$^{-1}$ in every case.

pressures may not be objectionable, when they occur under extreme conditions, where the known fundamental theories may fail in any case. However, we find for our model from (2.5), (2.11) and (2.14)

$$R^4 u = \text{const.} + \alpha\eta \int^t R^4 \rho_1 \, dt \tag{3.11}$$

that the radiation density is monotonically increasing and negative values are due to improper initial values as mentioned before in connection with (3.9).

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References

Ehlers, J. (1971). *General Relativity and Cosmology*. Academic Press, New York.
Hughston, L. P. and Shepley, L. C. (1970). *Astrophysical Journal*, **160**, 533.
Knight, Wayne R. (1972). Ph.D. Thesis, George Washington University.
May, T. L. and McVittie, G. C. (1970). *Monthly Notices of the Royal Astronomical Society*, **148**, 407.
McIntosh, C. B. G. (1968). *Monthly Notices of the Royal Astronomical Society*, **140**, 401.
Pegg, D. T. (1972). *Australian Journal of Physics*, **25**, 206.
Robertson, H. P. and Noonan, T. W. (1968). *Relativity and Cosmology*, p. 372. W. B. Saunders, Philadelphia.
Sciama, D. W. (1971). *Modern Cosmology*, p. 6. Cambridge University Press.
Stewart, John M. (1971). *Non-Equilibrium Relativistic Kinetic Theory*, Lecture Notes in Physics. Springer Verlag, Berlin-Heidelberg-New York.
Tolman, Richard C. (1931a). *Physical Review*, **37**, 1639.
Tolman, Richard C. (1931b). *Physical Review*, **38**, 797.
Tolman, Richard C. (1932). *Physical Review*, **39**, 320.
Tolman, Richard C. (1934). *Relativity, Thermodynamics and Cosmology*. Oxford University Press, London.